COORDINATE TRANSFORMATION IN THE DESCRIPTION OF PHYSICAL PHENOMENA

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ABSTRACT

The treatment of the collisions at secondary school level is a nice application of the general principles of mechanics, namely the conservation of the energy and the conservation of the momentum. By transforming the coordinates into a system moving with the centre of mass of the colliding bodies the use of quadratic equations can be avoided. Another interesting use of the coordinate transformations can be found in the theory of relativity. By the use of Lorentz-transformation between inertial frames it can be demonstrated that the quantum mechanical wave function is a result of the transformation of standing waves into the frame which is moving relative to the first one.

ELASTIC COLLISION

It is well known, that solving physical problems is simpler when an appropriate reference frame is used. It is suitable even in case of colliding bodies. Considering the elastic head-on collision of two discs of different masses we can use the laws of conservation of energy and momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2', \tag{1}$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 . \tag{2}$$

In these equations the masses m_i and the velocities v_i on the left side are known, and the value of velocities v_1' and v_2' after the collision are to be determined. Let the mass of the first and second discs be m_1 = 3kg and m_2 = 2kg, respectively. The first disc is moving to the right at velocity v_1 = 4m/s, while the second to the left at velocity v_2 = – 9m/s in the laboratory system which is attached to the table where the discs are moving. Substituting these values into the equations above, we get:

$$-6 = 3v_1' + 2v_2', \tag{3}$$

$$210 = 3v_1'^2 + 2v_2'^2 \tag{4}$$

Solving the equations, we get the velocities: v_1' = – 6,4m/s and v_2' = 6,6m/s after collision. But the calculation needs the solution of a quadratic equation which is beyond the curriculum of the 9th grade students.

However, the results can be obtained with simpler mathematical manipulations if the collision is described in the frame attached to the centre of mass of the system [1]. In this frame the centre of mass remains at rest during the collision and the sum of the momentums of the two bodies is zero all the time. It means that the momentum of the bodies is equal in magnitude but their direction is opposite. Since the kinetic energy of the system is also conserved, therefore only the direction of the momentums could be changed, the magnitudes of the momentums must remain the same during the collision.

The change between the frames happens on the basis of Galilean transformation, which leads to the classical addition formula of the velocities. In the laboratory frame of reference the centre of mass of the system is moving to the left with velocity V = –1,2m/s according to the formula

$$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} . \quad (5)$$

Now, let us calculate the momentums in the frame of reference which is moving at this velocity. It is the reference frame of the centre of mass. In this frame the velocity and momentum of the first body are 5,2m/s and 15,6kgm/s, respectively, while those of the second body are –7,8m/s, and –15,6kgm/s. As the reasoning given above shows, the direction of the momentums become opposite during the collision. From this the velocities of the first and second bodies can be calculated: – 5,2m/s and 7,8m/s, respectively. Transforming these velocities back to the laboratory system we get velocities v_1'= – 6,4m/s and v_2' = 6,6m/s. The result agrees with those gained previously.

Thanks to the appropriate choice of the frame the quadratic equations could be avoided. The train of thoughts based on the coordinate transformation is not an easy one, but according to my experiences it is comprehensible for 9th grade students.

THE DE BROGLIE WAVES

Derivation provided in this section helps in understanding the essence of the quantum mechanical wave function. Nowadays this procedure can be used probably only at a more advanced level of physics teaching (e.g. at introductory courses of universities.) However, my previous experiences obtained in teaching the elements of the theory of relativity at secondary school level indicate that in special secondary school classes the presented train of thought can be understandable.

In 1924, one of the keynotes of de Broglie’s theses was that a frequency can be assigned to the rest energy of micro-objects according to the relation

$$\upsilon_0 = \frac{m_0 c^2}{h} \quad (6)$$

(m_0 is the rest mass of a particle, c is the speed of light, h is the Planck constant) – see e.g. [2], [3]. At that time the idea was wondrous as this kind of frequency had come up before only in connection with differences of energy levels, and never in relation to the energy of a given state – see e.g. [4]. Nowadays, this relation is taken conventionally, since it is an organic part of quantum theory. In the abovementioned articles we can read about the relationship of the de Broglie waves and the theory of relativity in connection with de Broglie’s thesis. The question of de Broglie waves is also examined in [5]. Article [3] comes out from assuming that in the quantum’s rest frame a standing wave is present and the quantum is not point-like.

In the following the formula $\lambda = h/p$ will be derived from (6) with a self-constructed train of thought of the author of the present paper. It will be shown that if in the quantum's rest frame *K* a standing wave of *infinite* wavelength and rest-frequency m_0c^2/h is present, then using the Lorentz transformation we get in the frame *K'* a travelling wave with wavelength $\lambda' = h/(mV)$. This is the de Broglie wavelength. *V* is the relative velocity of frames *K* and *K'*, *m* is the relativistic mass of quantum.

It is known that the quantum in the rest frame has infinite de Broglie wavelength. Starting from this, the standing wave in the rest frame *K* can be represented with a function

$$y(x, t) = A \cdot \sin(\omega t)\,, \tag{7}$$

where the value of *y* does not depend on *x* – see Fig.1. The dependent variable *y* can be an arbitrary scalar physical quantity. The angular frequency in the phase is $\omega = 2\pi / T$.

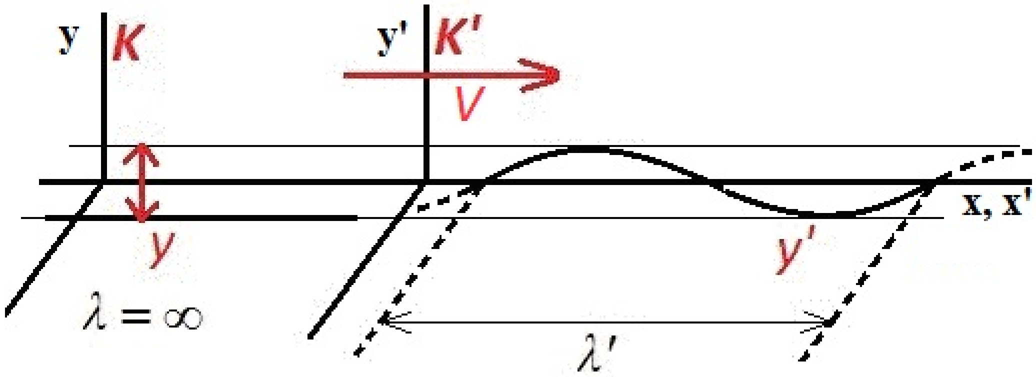

Fig.1. Function y(x, t) in the frame K and K'

The function takes a maximum value simultaneously at each point of frame *K*. Let us look at what we get if we consider the values of function in a system *K'* which moves parallel to the *x*-axis at a constant speed *V*. At this point, for describing the transformation of *x* and *t*, we have to use the formulas

$$x' = \frac{x - V \cdot t}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$t' = \frac{t - \frac{V \cdot x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{8}$$

of the Lorentz transformation.

Because of the relativity of simultaneity, we could also expect that while the maximum values of the function occur at each point of frame *K* at the same time, this will not be the same in frame *K'*. In a fixed moment *t'*, the value of the function is determined among others by coordinate *x'*.

In frame *K'* we have to work with a chosen coordinate *t'*. It results from the formula of time transformation that if we want to get the same coordinate *t'* at each point of the frame *K'*, we have to increase the *x* coordinate along with the coordinate *t*. More precisely, the increase of coordinate *t* with Δt can be compensated by increasing coordinate *x* with the suitable value of Δx. It is clear that in case of a fixed *x*, if Δt takes the time period from zero to *T*, then with a fixed *t*, the suitable Δx runs an interval related to a whole spatial period. Therefore, in frame *K'*, the state frozen at moment *t'* will be a harmonic function which is equal to such a 'crease' of the standing wave where the creases are characterized by wavelength λ', measured along the axis *x'*. We wish to determine this wavelength.

We transform the pair of events by data (*x*, *t*) and ($x + \Delta x$, $t + T$) into frame *K'*. The aim is to have a simultaneous pair of events in frame *K'*. According to the formula for time transformation, it is evident that the requirement *t'* = *const* refers to the relation

$$t - \frac{V \cdot x}{c^2} = const \qquad (9)$$

Therefore, the relation $T = V \cdot \Delta x / c^2$, otherwise $\Delta x = T\, c^2 / V$ must be valid. If this condition is valid, we will get a simultaneous pair of events in frame *K'* from the pair of events

$$(x, t) \text{ and } \left(x + \frac{Tc^2}{V},\ t + T\right) \qquad (10)$$

In frame *K'*, their distance is one period, which is equal to the sought wavelength *λ'*.

The event (*x*, *t*) is transformed to the coordinate

$$x_1' = \frac{x - V \cdot t}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad (11)$$

as the event ($x + T\, c^2 / V$, $t + T$) is transformed to the coordinate

$$x_2' = \frac{x - V \cdot t + \frac{Tc^2}{V} - V \cdot T}{\sqrt{1 - \frac{V^2}{c^2}}} = x_1' + \Delta x' \qquad (12)$$

In frame *K'*, the distance of these simultaneous events is

$$\Delta x' = \lambda' = \frac{\frac{Tc^2}{V} - V \cdot T}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad (13)$$

Based on this, we get the result

$$\lambda' = T\sqrt{1 - \frac{V^2}{c^2}} \cdot \frac{c^2}{V} \qquad (14)$$

Therefore, if we have a standing wave of *infinite* wavelength and angular frequency $\omega = 2\pi / T$ in frame *K*, we get a propagating wave of wavelength *λ'* and phase velocity of c^2 / V in frame *K'*. (As it is well known the phase velocity of a wave is the velocity of the propagation of a given vibration-state.)

Let us consider what happens in frame *K* if the frequency is equal to the rest frequency m_0c^2/h of a quantum. In this case $T = h / m_0c^2$ and we get the relation

$$\lambda' = \frac{h}{m_0c^2} \cdot \sqrt{1 - \frac{V^2}{c^2}} \cdot \frac{c^2}{V} = \frac{h}{mV}, \qquad (15)$$

with $m = m_0 \big/ \sqrt{1-V^2/c^2}$. This result is equal to the de Broglie wavelength in a general, relativistic case. Thus, the standing waves with *unlimited* wavelength (vibrations) are converted by Lorentz transformation into de Broglie waves.

We may apply a similar derivation when we would like to know the temporal periodicity in frame *K'*. In this case, we make the coordinate *x'* be fixed instead of *t'* in frame *K'*. This setting creates the condition $\Delta x = V \cdot \Delta t$ within which we have to choose the time span $\Delta t = T$.

Let us transform the events

$$(x, t) \text{ and } (x + V \cdot T \ , \quad t + T). \tag{16}$$

The time coordinate of event (x, t) in frame *K'* is

$$t_1' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \tag{17}$$

while the time coordinate of event $(x + V \cdot T , t+T)$ is

$$t_2' = \frac{t + T - \frac{V(x+V\cdot T)}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{18}$$

The difference between the two time coordinates is

$$T' = t_2' - t_1' = \frac{T - \frac{V^2 \cdot T}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{T \cdot (1-\frac{V^2}{c^2})}{\sqrt{1-\frac{V^2}{c^2}}} \tag{19}$$

based on that we can come to this relation:

$$T' = T\sqrt{1 - \frac{V^2}{c^2}}. \tag{20}$$

The same quantity is present in the relation derived for λ'. It refers to the fact that at the selected point, the vibrations happen with a reduced period *T'*, thus with an increased frequency of $\nu' = 1/T'$ in frame *K'*. The formula

$$\lambda' = \frac{T' c^2}{V} \tag{21}$$

shows that the vibrations in frame *K'* propagate as a wave of phase velocity

$$V_f = \frac{c^2}{V}. \tag{22}$$

Since $V < c$, the phase velocity is faster than the speed of light. The phase velocity depends only on relative velocity *V*. The velocity *V* of frame *K'*, which is compared to frame *K*, can be considered as the velocity of the quantum (in a sense, we can consider it as group velocity). According to the given relations, the product of phase velocity and group velocity is c^2. If relative velocity *V* approaches zero, the phase velocity will approach infinity.

The frequency of the de Broglie wave that we get in frame *K'* is

$$\upsilon' = \frac{\upsilon}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{23}$$

This frequency gives the total energy of the micro-object. If the rest mass, namely the rest frequency ν is doubled, the frequency ν' is duplicated too, and the equality

$$V_f = \frac{c^2}{V} = \lambda' \cdot \upsilon' \tag{24}$$

shows us that the de Broglie wavelength is also halved, since the phase velocity is constant. In frame *K*, only the rest energy $h \cdot \nu$ is present and it is related to the periodicity of time; on the other hand, in frame *K'*, there is a part of the total measurable energy $h \cdot \nu'$ which is connected with spatial periodicity (momentum *p*) and the periodicity of time. The disintegration of the total energy *E* into these two components is reflected in the formula

$$E^2 = m_0^2 c^4 + p^2 c^2. \tag{25}$$

The direction of momentum *p* is given by the direction of velocity *V* of frame *K* in frame *K'*.

CONCLUSIONS

The formula (14) derived for the wavelength in the moving frame *K'* provides the de Broglie wavelength if the rest frequency m_0c^2/h is substituted into it. Therefore it can be concluded that de Broglie waves are basically the relativistic transformations of standing waves with frequency m_0c^2/h. The relativity of simultaneity plays a crucial role in its derivation. Based on this the de Broglie wave cannot be understood outside the frame of relativity. This is not contradictory to the fact that these waves play a role in interference experiments, also in the case of small relative velocities.

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