PUZZLING PROBLEMS ON GRAVITY

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ABSTRACT

Although Newton's gravitational law is simple to state, it leads to a rich diversity of motions ranging from parabolic projectile motion to chaotic dynamics. The tools applied in these problems are also versatile. Here a bunch of puzzling problems on gravity is presented with the basic ideas of their solutions. Each problem requires a kind of unique, individual method and each solution teaches us something new. Some of the results are surprising. Most of the presented problems are used in the preparation for the International Physics Olympiad, and their solutions are attainable by elementary, secondary school methods.

INTRODUCTION

Throughout the history of physics the understanding of gravity went through several metamorphoses. The law of gravitational attraction was discovered in the early 17th century by Sir Isaac Newton, who has also realized that the same law governs the motion of a falling apple and the motion of the planets around the Sun. Later, at the end of the 19th century Roland Eötvös experimentally proved with high accuracy the equivalence of gravitational and inertial mass. In the beginning of the 20th century this fact became a cornerstone of Albert Einstein's general theory of relativity, which interpreted the gravitational interaction as the curvature of spacetime. Today, in modern physics we know that gravity is one of the four basic interactions of Nature.

In this work we are going to study gravity at the secondary school level, based on Newton's law of gravity. Apart from the first problem we restrict attention to regular planetary motions. The basic tools used in the solutions of the problems are Newton's gravitational law, Kepler's laws, conservation laws (energy, angular momentum, momentum), the geometry of conic sections, and in the last problem the theory of non-inertial reference frames.

In the first chapter five simpler problems are discussed, which are important either because of their final result or because of the methods used in the solution. In the 2nd and 3rd chapters single, more difficult problems are addressed. They are slightly beyond the secondary school level because of the mathematics and the abstractions used there.

Variants of some of the problems discussed here and other similar problems can be found in [1]. These problems are used in the preparation of the Hungarian team [2] for the International Physics Olympiad [3].

BASIC INSTRUCTIVE PROBLEMS

In this section we review some simple and instructive problems.

Problem 1 (Long pendulum): Find the period T of a mathematical pendulum whose length L is comparable to the radius R of the Earth. Assume that the angular deviation is small and that the mass of the pendulum is close to the surface of the Earth.

Solution: Let *m* be the mass of the pendulum. There are two forces acting on this mass; the gravitational force and the tension of the rope, as indicated in Fig.1.

In case of small angular deviations the magnitude of both forces is constant *mg*. In terms of the small angles α and β indicated in Fig.1, the equation of motion (in horizontal direction) has the form:

$$mL\ddot{\alpha} = -mgL(\alpha + \beta), \quad \text{where} \quad \alpha L = \beta R .$$

The solution of this equation is a simple harmonic motion with period $T = 2\pi\sqrt{\frac{LR}{g(R+L)}}$, which gives back the well-known formula for $L \ll R$.

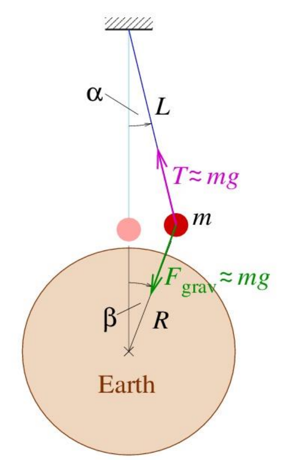

Fig.1. Forces acting on the long pendulum

Conclusion: The key point in the solution is that the magnitudes of the forces are constant (change only in second order of α) while their directions vary in first order of α. Generally, in the approximation of a vector field it is often useful to investigate separately the magnitude and direction of the vectors. This strategy is used also in the last problem.

The next problem is instructive for its own sake and it yields a result which can be used in other problems as well.

Problem 2 (Total energy of orbits): An object of mass *m* is orbiting another object of mass $M \ll m$. Express the total energy *E* in terms of the geometric parameters of the orbit. (These parameters are the semi-major axis *a*, the semi-minor axis *b* and the focal length *c*.)

Solution: First we assume that $E < 0$, so the orbit is an ellipse. Let r_P and r_A denote the distance of the perihelion *P* and aphelion *A* from the focal point at the central mass. The geometric relations between the distances indicated in Fig.2 are:

$$r_P = a - c , \qquad r_A = a + c , \qquad a^2 = b^2 + c^2 .$$

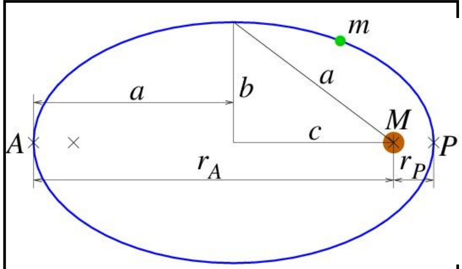

Fig.2. The elliptic orbit

The conservation of angular momentum and energy for the points *A* and *P* give the equations:

$$mv_A r_A = mv_P r_P , \qquad E_{ell} = \frac{mv_A^2}{2} - G\frac{mM}{r_A} = \frac{mv_P^2}{2} - G\frac{mM}{r_P} ,$$

where v_A and v_P are the speeds of the orbiting object at the points indicated in the subscripts. Eliminating the speeds and the focal length *c* from these equations, after a straightforward calculation the result

$$E_{ell} = -\frac{mMG}{2a} \qquad (1)$$

is obtained. (The negative sign indicates that the elliptic orbit is bounded.)

If $E > 0$ then the orbits are hyperbolas. The geometry of the hyperbola is not so familiar to secondary school students as that of the ellipse, so it is worth discussing it in detail. Fig.3 shows the hyperbola with its asymptotes and the lengths *a*, *b*, *c*. The aphelion is at infinity, so:

$$r_A = r_\infty = \infty , \qquad r_P = |PF| = c - a , \qquad c^2 = a^2 + b^2 .$$

Again we have to use the conservation of angular momentum and mechanical energy between the point at infinity and the perihelion P:

$$m v_\infty p = m v_P r_P \,, \qquad E_{hyp} = \frac{m v_\infty^2}{2} = \frac{m v_P^2}{2} - G\frac{mM}{r_P}\,.$$

Calculations similar to the elliptic case give the result:

$$E_{hyp} = \frac{mMG}{2a}\,. \qquad (2)$$

Conclusion: The total energy of the elliptic or hyperbolic orbits depends only on the semi-major axis a, and it is independent of the other parameters (b, c) of the orbit.

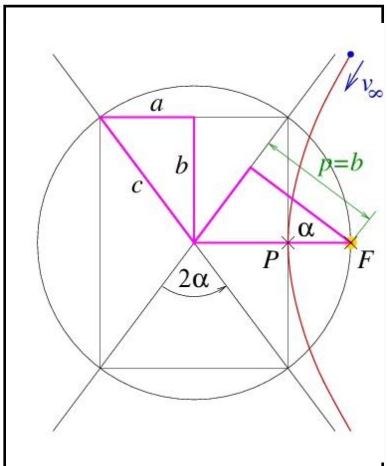

Fig.3. The hyperbolic orbit

Now we study in more detail another problem related to hyperbolic orbits.

Problem 3 (Deviation angle of hyperbolic orbits): A comet passes by the Sun. Determine its angle of deviation 2α in terms of the initial speed v_∞ (at infinity) and the impact parameter p (indicated in Fig.3).

Solution: With the use of the results of the previous problem the solution is simple. Using equation (2) and the formula $E_{hyp} = m v_\infty^2 / 2$, we get that $a = MG / v_\infty^2$. From Fig.3 it can be seen that $p = b$ and $\tan\alpha = a / b = \frac{MG}{p v_\infty^2}$.

Remark: Beside the conservation laws a key point of the solution is the relation $\tan\alpha = a / b$. This and the contents of Fig.3 should be discussed in more details in class [4].

Problem 4 (Racing satellites): Two satellites, A and B orbit the Earth on the same circular orbit, B lags behind A. How should B use its rocket in order to catch up with A? (Assume that the rocket can give only a quick impulse to the satellite.)

Solution: Our first, natural idea is to increase the velocity of B towards A. But this turns out to be wrong! Indeed, with this manoeuvre the total energy of the satellite is increased, so according to the result (1) of Problem 2, the semi-major axis a of the orbit increases. But due to Kepler's 3rd law, with increasing a the orbital period increases, too.

The above reasoning shows that paradoxically, the opposite manoeuvre has to be performed; the satellite should decrease its speed by giving an impulse opposite to its velocity. As a result of this, the satellite completes a faster cycle closer to the Earth, as indicated by the blue ellipse in Fig.4.

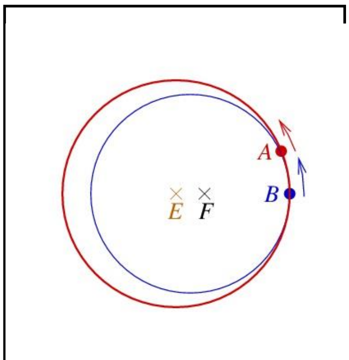

Fig.4. Satellite B decreases its speed

It is a nice exercise for practising first order approximations to find the relation between the change in the speed of the satellite Δv and the change in its period ΔT, provided that these quantities are small (relative to the total speed and total period, respectively). The result is:

$$\Delta T = \frac{6\pi R^2}{GM}\Delta v \qquad (3)$$

where M is the mass of the Earth and R is the radius of the circular orbit. The derivation of this formula is left to the interested students.

Problem 5 (Stopping the Moon): Imagine that the Moon's orbital motion around the Earth is suddenly stopped. How long would it take for the Moon to fall into the Earth? The orbital period of the Moon is T=28 days. (Assume that the Moon's orbit is a circle and neglect the Earth's motion around the Sun.)

Solution: The direct approach would be to solve the equation of motion but it is beyond the secondary school level.

A more tricky approach is to apply Kepler's 3rd law to compare the periods of the two different orbits of the Moon. The first orbit is the original circular orbit of radius R, period T=28 days and semi-major axis $a=R$. The second orbit is the degenerate ellipse corresponding to the motion of the Moon as it is falling into the Earth. The two foci of this ellipse are at the initial position of the Moon and at the Earth, so its semi-major axis is $a' = a/2$. Applying Kepler's 3rd law, the new period is $T' = T\sqrt{a'^3/a^3} = 2^{-\frac{3}{2}}T$. It means that the Moon would fall into the Earth in $T'/2 = 2^{-\frac{5}{2}}T = 4.95$ days.

ENVELOPING CURVE OF ORBITS

The problem discussed here is more difficult than the previous ones and it is for the best students.

Problem 6 (Enveloping curve of orbits): Let A be a fixed point in space at a distance d from a fixed sun S of mass M. Particles of mass m are shot from A in different directions at constant speed v. Which points can be reached by the particles? (Assume that v is small enough so the trajectories are ellipses.)

Solution: The arrangement has a rotational symmetry about the line AS, and all trajectories are planar curves, so it is enough to solve the problem in a single plane containing A and S. In this case our task is to determine the *enveloping curve* of a family of smooth curves.

Let us address this question generally. Let $\{C_\alpha\}_{\alpha\in I}$ be a family of smooth curves depending continuously on the real parameter α in the interval I, as shown in Fig.5. Pick two curves C_α and C_β corresponding to the parameter values α and β, and let K be their intersection point. It is heuristically clear from the figure that as $\beta \to \alpha$, the two curves come closer and closer to each other and their intersection K approaches a point of the enveloping curve, i.e.:

$$P_\alpha = \lim_{\beta\to\alpha} C_\alpha \cap C_\beta ,$$

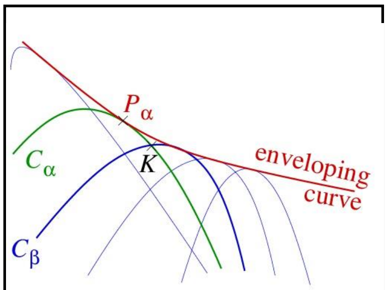

Fig.5. Enveloping curve of a family of curves

where P_α is the point where C_α touches the enveloping curve. So a general point of the enveloping curve is the intersection of two curves lying very close to each other.

Now we return to the original problem. Since the speed v and thus the total energy E of the particles shot in different directions is the same, due to the result (1) of Problem 2, the semi-major axis

$$a = -\frac{mMG}{2E} = \frac{MG}{2MG - v^2 d} d$$

of the orbits is constant as well. Let us consider two elliptic orbits C_α and C_β lying close to each other, as indicated in

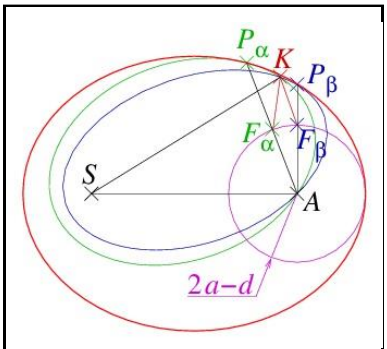

Fig.6. Two orbits (blue and green ellipses) touching the red enveloping curve

Fig.6. Let F_α, F_β denote their focal points (different from *S*), and let *K* be the intersection point of the two orbits (different from *A*). Since *A* is a point of both ellipses, $SA+AF_\alpha = SA+AF_\beta = 2a$, so $AF_\alpha = AF_\beta = 2a-d$. Thus the focal points F_α, F_β lie on a circle of centre *A*. Since *K* is also on both ellipses, $SK+KF_\alpha = SK+KF_\beta = 2a$, which means that $KF_\alpha = KF_\beta$. It means that in the limit $\beta \to \alpha$ the points *A*, $F_\alpha \approx F_\beta$ and $K \approx P_\alpha \approx P_\beta$ become collinear. Then for a general point P_α of the enveloping curve we have:

$$SP_\alpha + P_\alpha A = \underbrace{SP_\alpha + P_\alpha F_\alpha}_{2a} + \underbrace{F_\alpha A}_{2a-d} = 4a - d = \frac{2MG + v^2 d}{2MG - v^2 d} d,$$

so the enveloping curve is an ellipse of foci *A*, *S* and semi-major axis $2a - d/2$. Fig.7 shows a family of orbits which nicely fill out the enveloping ellipse.

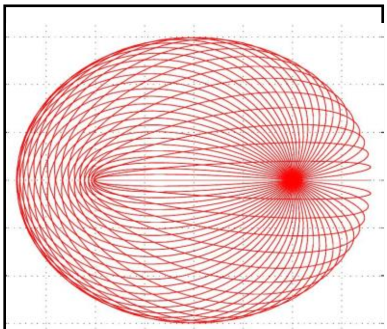
Fig.7. Many orbits fitting into the enveloping ellipse

Remark: Similar problems of finding the enveloping curves of certain trajectories can be formulated in optics (with light rays) and in hydrodynamics (with streams of fluids). The method discussed here helps in all cases.

MOTIONS OBSERVED FROM A SPACE STATION

In this section we discuss the motion of objects observed from a rotating reference frame. Non-inertial frames of reference and inertial forces are not involved in the Hungarian physics syllabus for secondary schools. The material of this section can be discussed in a special course for selected students after a systematic treatment of non-inertial reference frames and inertial forces [5].

Problem 7 (Motion around a space station): A space station is orbiting the Earth on a circular trajectory, facing always with the same side towards the Earth. A small object is thrown out of the space station with a small initial velocity **u**. How does the object move *relative to the space station*?

Solution: We solve the problem in the uniformly rotating reference frame of the space station. The axes are directed as indicated in Fig.8. The mass of the Earth and the small object are *M* and *m*, respectively. The radius of the orbit of the space station is denoted by *R*, and ω is the angular speed of the station. Furthermore, we shall use the constant $F_0 = GmM/R^2 = mR\omega^2$ to denote the magnitude of the centrifugal and the gravitational force acting on the small object in the space station.

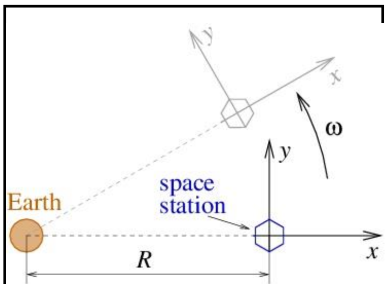

Fig.8. The reference frame of the space station

We expand all forces acting on the object in first order of the position $\mathbf{r} = (x, y, z)$ and the velocity $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ components of the small object. As we have seen in Problem 1, it is advantageous to expand first the magnitude of the forces and then the direction.

In first order the magnitude and the components of the gravitational force are:

$$F_g = \frac{GmM}{(R+x)^2 + y^2 + z^2} \approx F_0\left(1 - \frac{2x}{R}\right), \qquad \mathbf{F}_g \approx F_g \begin{bmatrix} -1 \\ -y/R \\ -z/R \end{bmatrix} \approx F_0 \begin{bmatrix} -1 + 2x/R \\ -y/R \\ -z/R \end{bmatrix}.$$

(We have used the fact that $\sin(\varepsilon) \approx \tan(\varepsilon) \approx \varepsilon$ and $\cos(\varepsilon) \approx 1$ for small angles ε.)

Similar expansions for the centrifugal force are:

$$F_{cf} = m\sqrt{(R+x)^2 + y^2}\,\omega^2 \approx F_0\left(1+\frac{x}{R}\right), \qquad \mathbf{F}_{cf} \approx F_{cf}\begin{bmatrix}1\\ y/R\\ 0\end{bmatrix} \approx F_0\begin{bmatrix}1+x/R\\ y/R\\ 0\end{bmatrix}.$$

Finally the Coriolis force is:

$$\mathbf{F}_{Cor} = -2m\boldsymbol{\omega}\times\mathbf{v} = -2m\begin{bmatrix}0\\0\\ \omega\end{bmatrix}\times\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = 2m\omega\begin{bmatrix}\dot{y}\\ -\dot{x}\\ 0\end{bmatrix}.$$

Substituting these expansions into the equation of motion $m\ddot{\mathbf{r}} = \mathbf{F}_g + \mathbf{F}_{cf} + \mathbf{F}_{Cor}$, after some simplification (cancellations) we get:

$$\ddot{x} = 2\omega\dot{y} + 3\omega^2 x, \qquad \ddot{y} = -2\omega\dot{x}, \qquad \ddot{z} = -\omega^2 z.$$

The last equation decouples from the other two and its solution is a harmonic motion with angular frequency ω in the *z* direction. Differentiating the first equation and using the second one, the equation $\ddot{v}_x = -\omega^2 v_x$ is obtained, whose solution is a similar harmonic motion. Taking into consideration the initial conditions $\mathbf{r}(0) = (0,0,0)$, $\mathbf{v}(0) = \mathbf{u} = (u_x, u_y, u_z)$, $\dot{\mathbf{v}}(0) = \mathbf{F}_{Cor}/m = 2\omega(u_y, -u_x, 0)$ we obtain the following solution:

$$\mathbf{r}(t) = \begin{bmatrix}x(t)\\ y(t)\\ z(t)\end{bmatrix} = \frac{1}{\omega}\begin{bmatrix}u_x\sin(\omega t) + 2u_y(1-\cos(\omega t))\\ 4u_y\sin(\omega t) + 2u_x(\cos(\omega t)-1) - 3u_y\omega t\\ u_z\sin(\omega t)\end{bmatrix}. \qquad (4)$$

Remark: It is worth discussing separately the special cases of the problem, when the initial velocity has only one non-zero component. It is also instructive to solve these special cases in the inertial reference frame of the Earth. The result (3) of Problem 4 can also be obtained from the general solution (4). We leave these investigations to the interested reader.

CONCLUSIONS

We have presented the solution of seven problems related to gravity, ranging from relatively simple ones to extremely difficult ones. Via these problems not only gravity, celestial mechanics can be taught to students, but many other things which are applicable in other branches of physics as well (e.g. approximation techniques, the geometry of conic sections, application of conservation laws, enveloping curves, non-inertial reference frames, differential equations, etc.). We hope that student readers enjoy learning physics from these nice problems and teacher readers think further some of the problems discussed here, and build into their own methodology some of the ideas presented here.

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