FUNNY MOTIONS OF BILLIARD BALLS

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ABSTRACT

The rolling and slipping motions of billiard balls on a horizontal surface are extensively studied in the literature. Most of these phenomena can be understood in the framework of high school physics. The variety of the possible motions and the difficulty of the physical ideas behind them make this topic interesting for a wide range of students from classroom physics to the level of International Physics Olympiad (IPhO). In this paper we present some interesting problems and examples related to the motions of billiard balls, which can be used in the preparation of talented students for international physics competitions such as IPhO and APhO.

INTRODUCTION

The detailed description of the various types of motion of billiard balls can be found in the literature. Gustave-Gaspard Coriolis, the famous French physicist was the first theorist who wrote a book about the subject in the 19th century [1]. Although Coriolis's calculations are based on Newtonian mechanics, for a high school student it is hard to follow the tedious explanations due to the complicated geometry of the three dimensional motions. Even Arnold Sommerfeld mentioned this topic in his famous books on theoretical physics [2]. Without the details he sketches a proof based on the rotational and translational equations of motion about the parabolic trajectory of the center of a billiard ball after a Coriolis-massé shot (see Problem 2). In a more recent book written by Alciatore [3] the Coriolis-massé shot aiming method is analyzed in more detail. In addition to the proof of the parabolic trajectory Alciatore presents a calculation about the final direction of motion of the ball. Although this derivation is surprisingly short and elegant, the effect of friction between the ball and the table during cue stick impact is completely neglected. However, as Coriolis showed this friction has no effect on the final cue ball direction.

In the following discussion we want to show that the essence of these phenomena can be understood on a high school level. We discuss two different situations (Problem 1 and Problem 2) in which the application of the conservation of angular momentum provides a simple and elegant way of solution. We have used these problems in the last couple of years during the Hungarian preparation courses for the International Physics Olympiad. According to our experience, these kind of problems help the students deepen their knowledge and understanding about angular momentum and rotational motion. The problems presented here can be found in the problem collection written by the authors [4].

PROBLEM 1: MOTION ALONG A LINE

Problem 1. A ball, initially at rest on a billiard table, is struck by a cue tip at the point *T* shown in Fig.1. The cue lies in the vertical plane containing *T*, the centre of the ball *C*, and the ball's point of contact with the table *P*; consequently, so does the line of action of the resulting impulse. Construct the direction in which the cue should be aligned in order that after the shot, the ball's subsequent rotational and slipping motions terminate at the same instant and the ball comes to a halt. (As a result of chalking of the cue tip, the coefficient of friction between it and the ball is sufficiently large that there is no slippage between them during the cue stroke.)

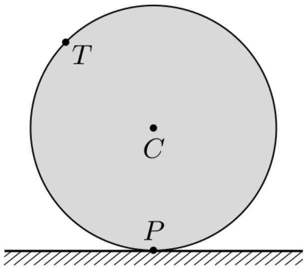

Fig.1. Position of the point of contact (P), the center of the ball (C) and the hitting point (T)

Solution 1. In general, after receiving an impulse from the cue, the billiard ball both rolls and slips and the instantaneous speed of its point of contact with the table is not zero. This 'grating' continues until, as a result of kinetic frictional forces, the velocity of that point relative to the table decreases to zero; after that, the ball continues to roll but without slipping.

Consider the point *P* at which the ball touches the table before the shot is taken. Note, that *P* denotes a *fixed* point *on the table*, and not the current contact point of the ball and table (which accelerates, or decelerates, during the stroke and the subsequent 'grating'). The total angular momentum of the ball about this point is zero before the shot, as well as at the simultaneous end of the rolling and slipping motions (when the ball again becomes stationary).

During the motion that follows the cue stroke, the net torque about *P* of the forces acting on the ball is zero, because the gravitational force and the normal reaction of the table cancel each other, and the line of action of the frictional force always passes through *P*. The angular momentum of the ball about *P* can only remain at zero throughout (from before the stroke until after the final halt) if it does not receive any during the stroke itself; this requires that the line of action of the impulse, and hence that of the cue, must be directed through point *P* (see Fig.2).

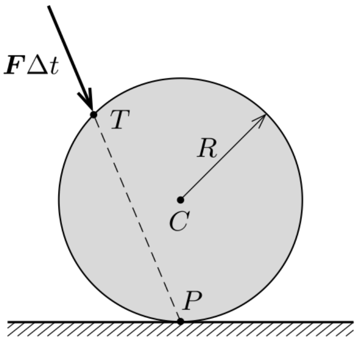

Fig.2. The direction of the cue must go through the point *P*

PROBLEM 2: MOTION IN 3D – THE 'CORIOLIS-MASSÉ' SHOT

Problem 2. If the line of action of the impulse in *Problem 1* does not lie in the vertical plane defined by the points *T*, *C* and *P*, then, just after the shot, the ball's angular velocity vector will not be perpendicular to the velocity of its centre of mass. Billiard players call this shot a *Coriolis-massé*.

Such a shot is shown in Fig.3, in which the line of action of the impulse meets the ball's surface (for a second time) at *T'* and the table at *A*.

a) What kind of trajectory does the ball's centre of mass follow from just after the shot until the point at which simultaneous rolling and slipping ceases?

b) In which direction, relative to the line *PA*, will the ball continue its path once it starts to roll without slipping?

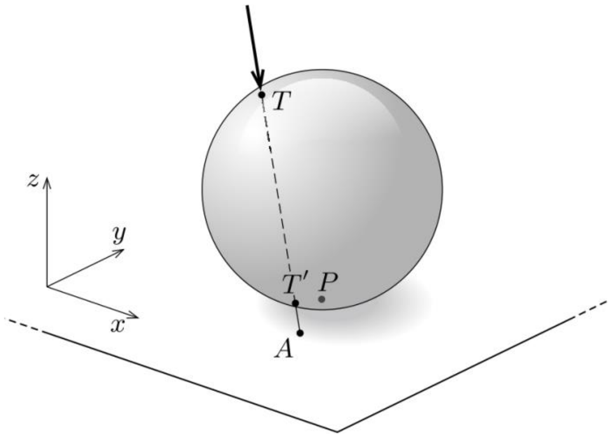

Fig.3. The direction of impulse in Problem 2.

(Assume that, whatever the downward force acting on it, the billiard cloth does not 'become squashed', and the ball's contact with it is always a point contact.)

Solution 2. Denote the vector pointing from the centre *C* of the billiard ball to its lowest point (where it touches the table) by $\boldsymbol{R}$, the mass of the ball by m, the velocity of its centre of mass by $\boldsymbol{v}$, and its angular velocity by $\boldsymbol{\omega}$**.**

As noted in the problem, for a general *Coriolis-massé* $\boldsymbol{\omega}$ will not be perpendicular to $\boldsymbol{v}$, and so the velocity of the lowest point of the ball,

$$\boldsymbol{v}_P = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{R},$$

will not be parallel to the velocity of the centre of the ball, even at the start of the motion. A similar connection holds between the corresponding accelerations and the angular acceleration:

$$\dot{\boldsymbol{v}}_P = \dot{\boldsymbol{v}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{R}. \qquad (1)$$

During the 'grating' motion, the horizontal acceleration of the ball and its angular acceleration are both caused by the frictional force $\boldsymbol{F}$, and so the dynamical equations for the translational and rotational motion can be written as follows:

$$\boldsymbol{F} = m\dot{\boldsymbol{v}},$$

$$\boldsymbol{R} \times \boldsymbol{F} = \frac{2}{5} mR^2 \dot{\boldsymbol{\omega}}.$$

Inserting expressions for $\dot{\boldsymbol{v}}$ and $\dot{\boldsymbol{\omega}}$, obtained from these two equations, into equation (1):

$$\dot{\boldsymbol{v}}_P = \frac{1}{m}\boldsymbol{F} + \frac{5}{2mR^2}(\boldsymbol{R} \times \boldsymbol{F}) \times \boldsymbol{R}.$$

Now $\boldsymbol{F}$ and $\boldsymbol{R}$ are necessarily mutually perpendicular, and so using either the right-hand rule or the vector triple product identity, it follows that

$$(\boldsymbol{R} \times \boldsymbol{F}) \times \boldsymbol{R} = R^2\boldsymbol{F}.$$

So finally we have that

$$\dot{\boldsymbol{v}}_P = \frac{7}{2m}\boldsymbol{F}. \qquad (2)$$

The magnitude of the kinetic frictional force is μmg (where μ is the coefficient of friction), and its direction is opposed to that of the velocity of the lowest point of the ball:

$$\boldsymbol{F} = -\mu mg\frac{\boldsymbol{v}_P}{|\boldsymbol{v}_P|}. \qquad (3)$$

Combining this with equation (2), we have

$$\dot{\boldsymbol{v}}_P = -\frac{7}{2}\mu g\frac{\boldsymbol{v}_P}{|\boldsymbol{v}_P|}. \qquad (4)$$

Equation (4) shows that the velocity of the ball's lowest point has a constant direction throughout the simultaneous rolling and slipping motion, and that its magnitude decreases uniformly to zero at a rate of $-(7/2)\mu g$. It then follows from (3) that not only the magnitude of the frictional force, but also its direction, is constant. As this direction does not coincide with that of the initial velocity of its centre of mass, the billiard ball moves along a *parabolic* (rather than a straight) trajectory (see Fig.4).

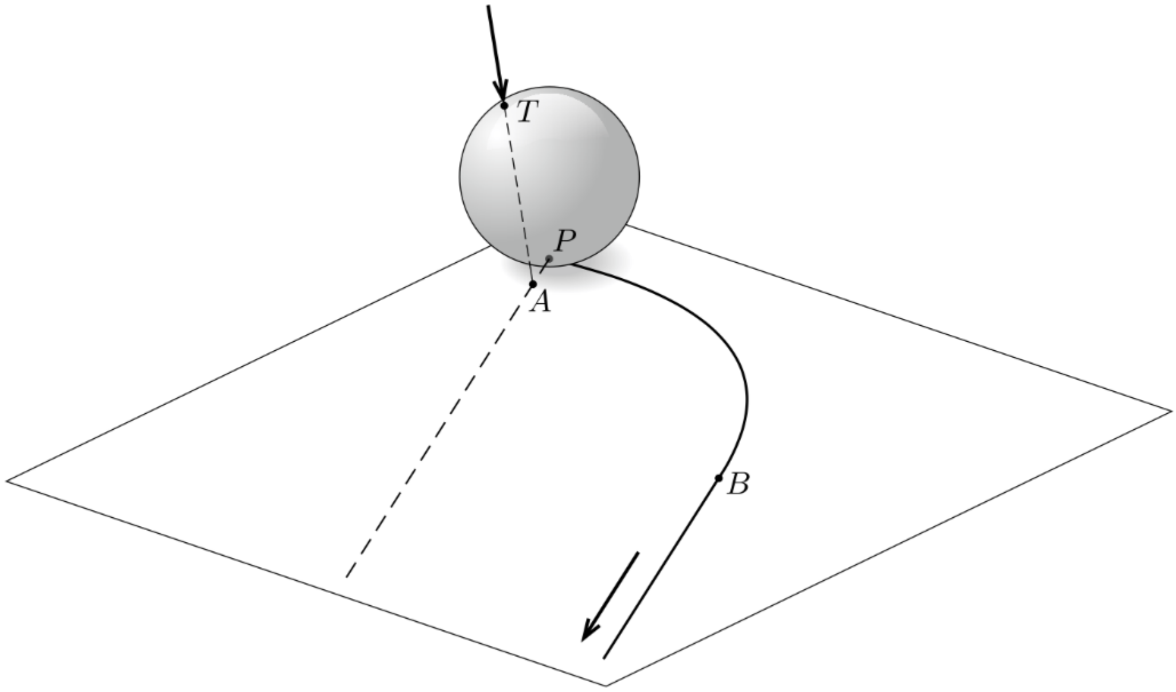

Fig.4. The parabolic trajectory of the center of the ball in the case of a Coriolis-massé shot

When the velocity of the lowest point of the ball becomes zero (this happens at *B* in Fig.4), the ball continues to roll, but without any slipping, until air drag and rolling friction bring it to a halt. Its straight-line path is along the tangent to the parabola at point *B*.

b) The final direction of the ball's motion can be found with the help of the law of conservation of angular momentum. We investigate the angular momentum of the ball about the line *PA*.

Angular momentum is a vector quantity, which is defined relative to a fixed (but arbitrarily chosen) *point* in space. But, it is also the case that a component of angular momentum in a given direction can be defined by an *axis* which lies in that direction. In this problem, for example - as will be shown later - the angular momentum of the ball relative to the *point P* is not conserved, but the component of angular momentum parallel to the line *PA* does remain constant.

Initially, the ball is at rest, so its angular momentum is zero. During the short time interval of the shot, the lines of action of the forces acting on the ball (the force of the shot exerted by the cue, the normal reaction force of the table, the frictional force, and the gravitational force) all pass through various points on the line *PA*. So, just after the shot, the angular momentum component defined by this line is also zero. This situation does not change as the ball moves along the parabolic arc *PB*, because the gravitational force and the normal reaction of the table cancel each other out, and the torque about this axis due to the frictional force is always zero (since the force and the axis lie in the same plane).

So, on the one hand, after finishing the 'grating' section of the motion the angular momentum vector of the ball remains constant -- it is horizontal, and perpendicular to the velocity of the centre of mass. But, on the other hand, as we have just shown, its component parallel to the line *PA* is zero. There is only one way to reconcile these two conclusions, and that is that the ball's path is *parallel* to the line *PA*.

CONCLUSIONS

In this paper we presented two sample problems which can be used to illustrate the usefulness of conservation of angular momentum when describing the quite complicated motion of billiard balls on a horizontal surface. Problems like these can be used for probing and improving the creative physical thinking of the gifted students, so such exercises could help the preparation of pupils for international physics competitions (such as APhO and IPhO) for high school students.

ACKNOWLEDGMENTS

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