Teaching Physics Innovatively 2015.

Puzzling problems on gravity

Tamás Tasnádi [1]

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[1] BME, Institute of Mathematics

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Overview

Simple laws $\Longrightarrow$ complicated motions

Toolkit:

- Newton's gravitational law
- Kepler's laws
- Conservation laws (energy, angular momentum, momentum)
- Geometry of conic sections

Most of these problems are discussed in the courses preparing for the IPhO.

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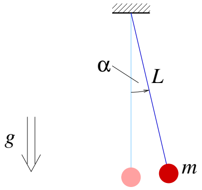
Long pendulum

Problem: Find the period T of a mathematical pendulum whose length L is comparable to the radius of the Earth R. (Assumptions: small angular deviation; small distance from the Earth.)

Solution:

- $|\mathbf{T}| \approx |\mathbf{F}_{\text{grav}}| \approx mg$
- $\mathbf{T}$ and $\mathbf{F}_{\text{grav}}$ are not vertical
- Equation of motion ($\alpha, \beta \ll 1$):

$$mL\ddot{\alpha} = -mgL(\alpha + \beta), \quad \alpha L = \beta R$$



- Result:

$$T = 2\pi\sqrt{\frac{L}{g}\frac{R}{R+L}} \quad \xrightarrow{\frac{R}{L}\to\infty} \quad 2\pi\sqrt{\frac{L}{g}}$$

Conclusion: Approximate separately the magnitude and the direction of a vector(field).

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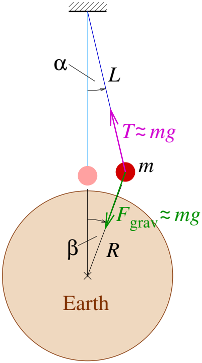

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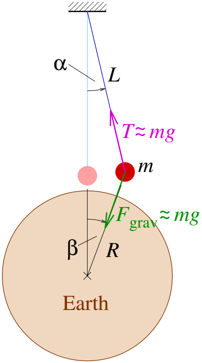

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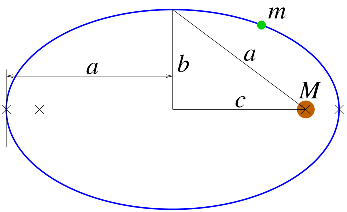
Total energy of elliptic orbits

Problem: An object of mass m is orbiting another object of mass $M \gg m$. Express the total mechanical energy $E(a, b)$ in terms of the major and minor axes a and b.



Solution:

- Distance of perihelion P and aphelion A:
  $r_P = a - c, \qquad r_A = a + c, \qquad a^2 = b^2 + c^2$
- Energy conservation: $E(a, b) = \frac{mv_P^2}{2} - G\frac{mM}{r_P} = \frac{mv_A^2}{2} - G\frac{mM}{r_A}$
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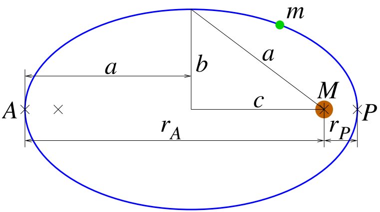

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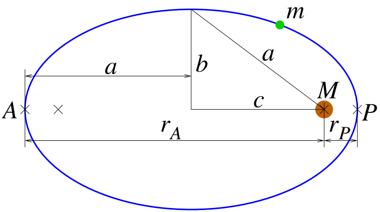

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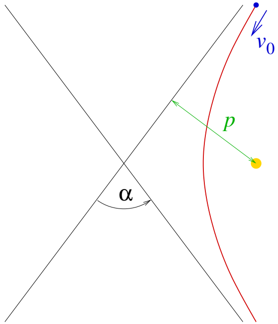
Deviation angle of hyperbolic orbits

Problem: A comet passes by the Sun. Determine its angle of deviation α in terms of the initial speed v_0 and the impact parameter p.



Hint:

- Apply energy and angular momentum conservation for the perihelion P and the point at infinity
- Use the geometry of the hyperbola:

$$c^2 = a^2 + b^2, \quad \tan\left(\frac{\alpha}{2}\right) = \frac{b}{a}, \quad p = a, \quad PF = c - b$$

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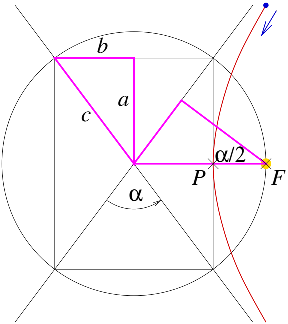

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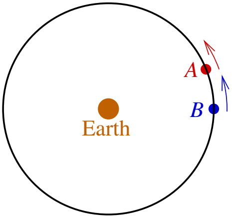
Racing satellites

Problem: Two satellites, A and B orbit the Earth on the same circular orbit, B lags behind A. How should B use its rocket in order to catch up with A? (Assume that the rocket can give only a quick impulse to the satellite.)



Solution:

- If B increases its speed $\Longrightarrow E(a) = -\frac{mMG}{2a} < 0$ increases $\Longrightarrow$ a increases $\Longrightarrow$ period T increases WRONG!!!
- If B decreases its speed $\Longrightarrow E(a)$ decreases $\Longrightarrow$ a decreases $\Longrightarrow$ T decreases CORRECT!!!

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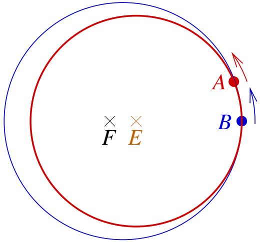

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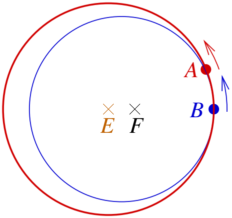

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Stopping the Moon

Problem: Imagine that the Moon's orbital motion around the Earth is suddenly stopped. How long would it take for the Moon to fall into the Earth?

Remark: The direct integration is beyond the secondary school level.

Idea: Apply Kepler's third law for the two orbits of the Moon.

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Motions observed from a space station

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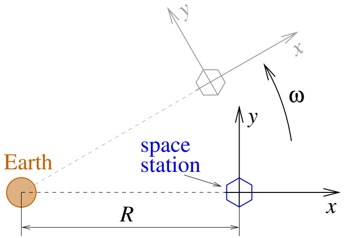

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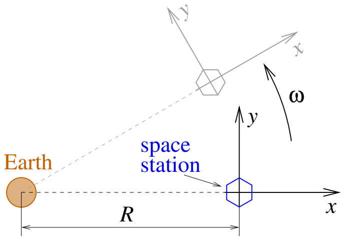

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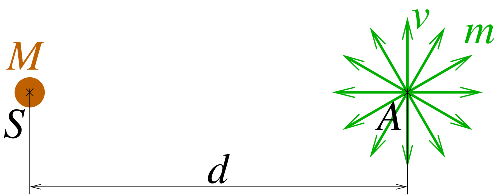
The problem

Let A be a fixed point in space at a distance d from a fixed sun S of mass M. Particles of mass m are shot from A in different directions at constant speed v. Which points can be reached by the particles?



The enveloping curve of the orbits is to be found.

First notices

- Planar trajectories, rotational symmetry about $AS \implies$ restrict attention to the plane
- Total energy: $E = \frac{1}{2}mv^2 - G\frac{mM}{d}$ is constant
- If $E \geq 0 \implies$ infinite orbits $\implies$ any point can be reached (proof is omitted now)
- If $E < 0 \implies$ finite orbits $\implies$ only a bounded region can be reached (this case is studied now)
- The semi-major axis a of the orbits:

$$E = \frac{1}{2}mv^2 - G\frac{mM}{d} = -\frac{mMG}{2a} \quad \implies \quad a = \frac{MG}{2MG - dv^2}d$$

- $2a > d$

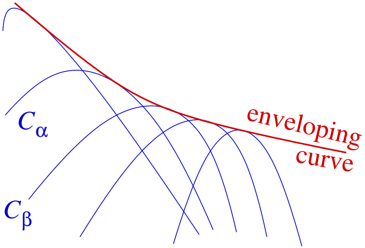
Enveloping curve

Question: How to obtain the enveloping curve of a family of curves $\{C_\alpha\}_{\alpha \in I}$?

Answer:

- Two curves close to each other: C_α and C_β
- The intersection K of C_α and C_β is close to the enveloping curve
- The point P_α where C_α touches the enveloping curve is:

$$P_\alpha = \lim_{\beta \to \alpha} \underbrace{C_\alpha \cap C_\beta}_{K}$$



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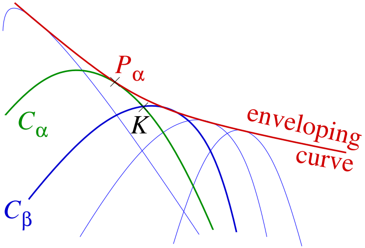
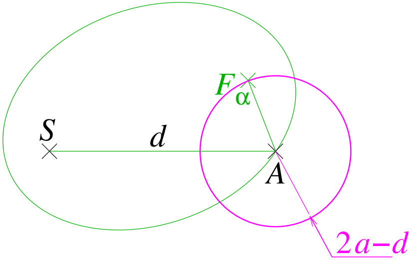


Solution (Enveloping curve of elliptic orbits)

- Focal points of an orbit: S and F_α
  $AS + AF_\alpha = 2a \Longrightarrow AF_\alpha = 2a - d$ (circle)
- $SK + KF_\alpha = SK + KF_\beta = 2a \Longrightarrow KF_\alpha = KF_\beta$
  $\Longrightarrow A$, F_α, P_α are collinear
- $SP_\alpha + P_\alpha A = \underbrace{SP_\alpha + P_\alpha F_\alpha}_{2a} + \underbrace{F_\alpha A}_{2a-d} = 4a - d$; ellipse



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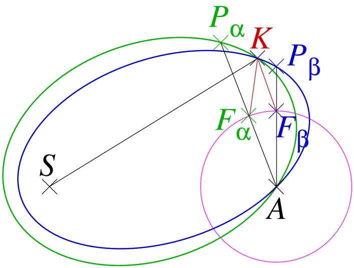

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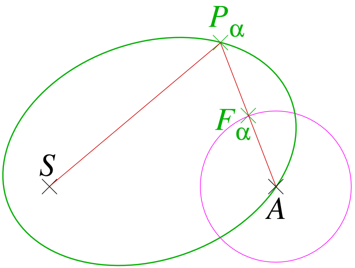

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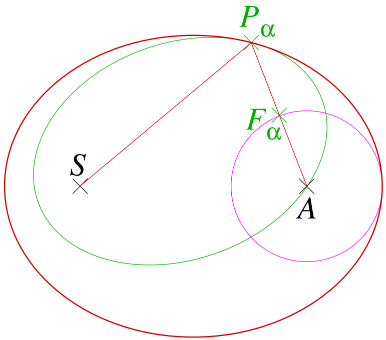
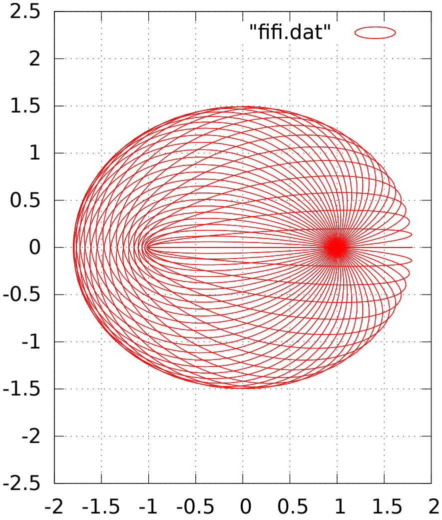


Enveloping curve is an ellipse of foci S and A, semi-major axis $2a - \frac{d}{2}$.

The orbits



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